5.1 微分法

(Differentiable) $f:\mathbb{R}^m\to\mathbb{R}^n$ **differentiable at** $a\iff\exists$**linear mapping** $A:\mathbb{R}^m\to\mathbb{R}^n\ s.t.\ f(a+h)=f(a)+Ah+o(||h||)$

Proposition:

$(f\pm g)'=f'\pm g'$, etc.

Chain Rule.

Inverse Function: $f:\mathbb{R}\to\mathbb{R}$ differentiable bijection, thus f^{-1} is well-defined. If $f^{-1}(x)$ is differentiable at the point x.(This is important! Previous conditions cannot guarentee the existence of the derivation: $f(x)=x^3$)

We have $(f^{-1}(x))'=\dfrac{1}{f'(f^{-1}(x))}$

(Local Max/Min) $f\ (a,b)\to\mathbb{R}$**, call** f **reaches local maximum/minimum at point** $x\in(a,b)$**, if exists** $B_r(x)\ s.t.\ \forall y\in B_r(x)\cap(a,b), f(y)\le f(x)(\ge)$

Similarly we have the definition of global maximum/minimum.

Prop.(Fermat's Lemma) $f:(a,b)\to\mathbb{R}$ differentiable and x is a local maximum point, then $f'(x)=0$.

(Rolle mean value theorem) $f:[a,b]\to\mathbb{R}$ **continuous, differentiable on** (a,b) **and** $f(a)=f(b)$**, then exists** $\xi\in(a,b), f'(\xi)=0$**.**

Similarly we get a wider version.

(MVT) $f,g:[a,b]\to\mathbb{R}$ **continuous, differentiable on** (a,b)**,**
$\exists\xi\in(a,b)\ s.t.\ \dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(\xi)}{g'(\xi)}(g(a)\neq g(b))$

(Take $g(x)=x$ we get Lagarange MVT)

Corollary: $f:\mathbb{R}\to\mathbb{R}$ differentiable, $f'\equiv 0 \iff f$ constant.

Prop. **(L'Hopital)** $f,g:(a,b)\to\mathbb{R}$ **differential and** $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}=A$

If: 1. $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ **Or 2.** $\lim_{x\to a} g(x)=\infty$

Then $\lim_{x\to a}\dfrac{f(x)}{g(x)}=A$**.**

(Taylor expansion) $f:(a,b)\to\mathbb{R}$ **is n-th differentiable,**
$f(x)=\sum_{k=0}^{n}\dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\dfrac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1},\ (\xi\in(x,x_0))$

Proof: We construct the function $g(x)=\sum_{k=0}^{n}\dfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k+M(x-x_0)^{n+1}$,
and let $g(x)=f(x), g(x_0)=f(x_0), thus\ \exists\xi\ s.t.\ g'(\xi)=f'(\xi)$

Use induction we get the desired result.

6.3 PDE

1. 热方程

$f:\mathbb{R}_t\times\mathbb{R}^3\to\mathbb{R}$

满足$\partial_t f-\triangle f=0$，($\triangle f=\sum\partial^2_{x_ix_i}f=\nabla(\nabla f)$)

假定：f代表温度；热流量$\vec{J}=-\kappa\nabla f$ (Fourier's law)；能量守恒
$\partial_t(\int_U f dt)=-\int_{\partial U}\vec{J}\cdot\vec{n}$（区域能量损失等于通过边界的热流量）

而由Stokes' 定理$\int_{\partial U}F\cdot\vec{n}=\int_U\nabla f$

于是$\int_U\left[\partial_t f-\nabla\cdot(\kappa\nabla f)\right]=0\implies\partial f=\kappa\triangle f$

2. 调和方程

即考虑热方程的稳态，此时$\triangle f=0$

(Poisson 方程：外加热源但是仍然进入稳态：$\triangle f=Const.$)

3. 波方程

$\partial_{tt}u-\triangle u=0$

现在只考虑弦的震动，$\partial_{tt}u$表示加速度，

8.2 ODE; Picard-Lindelof-Cauchy-Lipschitz Theorem

（Picard-Lindelof）$D\subseteq\mathbb{R}\times\mathbb{R}^n, f:D\to\mathbb{R}^n$关于$t$连续，关于$x$Lipschitz在$[t_0-a,t_0+a]\times B_r(x_0)$上成立。那么存在$a'\geq 0$，使得方程$x'(t)=f(t,x),x(t_0)=x_0$存在唯一的$[t_0-a',t_0+a']$上的$C^1$解。

证：

记$X=\{f\in C(I,\mathbb{R}^n),I\in t_0(to\ be\ fixed),f(t_0)=x_0\}$，并且配备上确范数诱导的度量。它是$C(I)$的闭子集，于是自然完备。

考虑X上算子$T:X\to X:y(t)\mapsto z(t)$

$z(t)=y(t_0)+\int_{t_0}^t f(s,y(s))\,ds$，那么$T(x)=x\iff Desired\ Solution.$

只需验证T满足压缩映射：

$$\begin{aligned}\sup||T(y)-T(z)||&=\sup||\int_{t_0}^t f(s,y(s))-f(s,z(s))\,ds||\\&\leq\sup\int_{t_0}^t L|y(s)-z(s)|\,ds\leq La'\sup||y(s)-z(s)||\end{aligned}$$

取a'充分小即可满足压缩映射，并且满足y,z不超过$[t_0-a,t_0+a]\times B_r(x_0)$。

8.3 反函数定理

反函数定理：$F:U\to\mathbb{R}^n$，Open Set $U\subseteq\mathbb{R}^n$。$F\in C^1(U)$，$DF(x_0)$ 可逆（其中$x_0\in U$）

那么存在$V\ni x_0,W\ni f(x_0)$。使得$F|_V:V\to W$是双射，并且反函数$F^{-1}:W\to V$也是C^1的。

单位扰动定理：X是Banach空间，$\Phi:X\to X$，记$\Psi(x)=\Phi(x)-x$。

如果在邻域$\overline{B_r(x_0)}$上有：$||\Psi(x_1)-\Psi(x_2)||\leq\gamma||x_1-x_2||$，$(0\leq\gamma<1)$

那么对于$y \in \overline{B_{(1-\gamma)r}(\Phi(x_0))}$，$y$在$\overline{B_r(x_0)}$中有唯一的原像。

证：

固定y，希望找到$x\ s.t.\ \Phi(x) = y \iff x + \Psi(x) = y \iff x = y - \Psi(x)$

因此考虑映射$T : t \mapsto y - \Psi(t) : B_r(x_0) \to B_r(x_0)$

（值域为$B_r(x_0)$因为：

$$\begin{aligned}||y - \Psi(t) - x_0|| \le ||y - \Phi(x_0)|| + ||\Phi(x_0) - \Psi(t) - x_0|| &\le (1-\gamma)r + ||\Psi(x_0) - \Psi(t)|| \\ &\le (1-\gamma)r + \gamma r = r\end{aligned}$$

）

$$||T(x_1) - T(x_2)|| = ||\Psi(x_1) - \Psi(x_2)|| \le \gamma||x_1 - x_2||$$

于是由Banach不动点定理，结论自明。

回到反函数定理：

由于$F(x + x_0) - F(x_0) = DF(x_0)x + o(||x||)$，

$(DF(x_0))^{-1}(F(x + x_0) - F(x_0)) = x + o(||x||)$

因此（为了运用单位扰动定理），取

$$\begin{aligned}\Phi(x) = (DF(x_0))^{-1}(F(x + x_0) - F(x)) &= (DF(x_0))^{-1}(F(x + x_0) - F(x_0) - DF(x_0)x + D \\ &= x + (DF(x_0))^{-1}(F(x + x_0) - F(x_0) - DF(x_0)x) \\ &= x + \Psi(x)\end{aligned}$$

Check Conditions.

$$\begin{aligned}||\Psi(x_1) - \Psi(x_2)|| &= ||(DF(x_0)^{-1})(F(x + x_1) - F(x + x_2) - DF(x_0)(x_1 - x_2))|| \\ \le ||DF(x_0)^{-1}||_{matrix\ norm}|| \int_0^1 &\left[DF(x_2 + t(x_2 - x_1) \cdot (x_2 - x_1) - DF(x_0) \cdot (x_2 - x_1)\right] dt||\end{aligned}$$

可以寻找充分小的r，使得$||DF(x_2 + t(x_2 - x_1)) - DF(x_0)||$较小，从而满足单位扰动定理条件）

那么对于每个$y \in B_{(1-\gamma)r}(x_0)$，存在唯一的$x \in B_r(x_0)$，$\Phi(x) = y$。

即$(DF(x_0))^{-1}(F(x + x_0) - F(x)) = y \implies F(x_0 + x) = F(x_0) + DF(x_0)y$

即对于集合$\{F(x_0) + DF(x_0)y | y \in B_{(1-\gamma)r}(x_0)\}$中的元素都能找到原像。

记反函数为G，那么：

$$
\begin{aligned}
||G(y_1)-G(y_2)|| = ||x_1-x_2|| &\le ||(x_1+\Psi(x_1))-(x_2+\Psi(x_2))||+||\Psi(x_1)-\Psi(x_2)|| \\
&\le ||y_1-y_2||+\gamma\cdot||x_1-x_2||
\end{aligned}
$$

于是$||x_1-x_2|| \le \dfrac{1}{1-\gamma}||y_1-y_2||$，从而说明了连续性。

10. Fouriér Analysis

10.0 Preliminaries

Prop. $1. f*g=g*f; 2. f*g\ continuous\ (as\ long\ as\ f,g\ integrable)$

Prop2.Pf.

若g连续，
$$
\begin{aligned}
&(f*g)(x+\Delta x)-(f*g)(x) \\
&=\int_{-\pi}^{\pi} f(y)[g(x+\Delta x-y)-g(x-y)]\mathrm{d}y \to 0\ as\ \Delta x\to 0
\end{aligned}
$$

（这个证明不够强，但是指出卷积函数只要一个具有比较好的性质：连续、可微、光滑etc.，都能够保证卷积后的结果继承了这一性质）

Lem. 给定一个可积函数g，它当然有界：$|g|\le B$，则存在一族连续函数g_k，$\sup|g_k|\le B, \int_{-\pi}^{\pi}|g_k-g|\mathrm{d}x\to 0$（很容易联想到简单逼近）

Lem.Pf.

取$g_k(x)=\sup\limits_{[m/k,(m+1)/k]\ni x} f(x)$，$g_k$满足条件，尽管不连续。

对g_k的间断点做插值，插值带来的误差是可以控制的：每段在端点处用$1/k^2$的长度做插值，带来的误差被$k\times 1/k^2\times B$控制。

取定g_k，
$$
\begin{aligned}
&|(f*g)(x+\Delta x)-(f*g)(x)| \\
&\le |(f*g)(x+\Delta x)-(f*g_k)(x+\Delta x)| \\
&+|(f*g_k)(x+\Delta x)-(f*g_k)(x)|+ \\
&|(f*g_k)(x)-(f*g)(x)|
\end{aligned}
$$

第三项$|(f*g_k)(x)-(f*g)(x)|\le\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}|f(y)||g_k(x-y)-g(x-y)|$从而被$k$控制。第一项同理，第二项被$\Delta x$控制，于是得证。

（或：$f*g_k\rightrightarrows f*g$，即说明连续）

10.1 Fouriér Series

Definiton: $f:\mathbb{R}\to\mathbb{R}, 2\pi-$periodic. $[-\pi,\pi]$上Riemman可积。

$$\hat{f}(n):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\mathrm{d}x,\ f(x)\sim\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{inx}$$

（$\hat{f}(n)$的选取动机是f在基e^{inx}上的投影，其中配备的内积为卷积）

这里：右侧求和定义为对称求和主值$\lim\limits_{k\to\infty}\sum\limits_{n=-k}^{n=k}\hat{f}(n)e^{-inx}$.

Examples. Given $f(x)=\begin{cases}1 & [0,\pi)\\ -1 & [-\pi,0)\end{cases}$,

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)(\cos nx-i\sin nx)\mathrm{d}x=-\frac{i}{\pi}\int_0^{\pi}\sin nx\mathrm{d}x$$
$$=\frac{[(-1)^n-1]i}{n\pi}(n\ne 0)$$

$n=0,\hat{f}(n)=0$

于是$\sum\limits_{n\in\mathbb{Z}}\dfrac{i[(-1)^n-1]}{n\pi}e^{inx}$在$x=0$时恒为0，自然不逐点收敛到函数$f$。

接下来对这一级数的收敛性进行讨论：

正如对称主值那样，记$S_Nf(x)=\sum\limits_{n\le|N|}\hat{f}(n)e^{inx}$，在何时$S_nf(x)\to f(x)$. 是否能更进一步做到$\rightrightarrows$? (Fejér Thm.)

$$S_N f(x) = \sum_{n\le|N|} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{in(x-y)}\mathrm{d}y = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n\le|N|} e^{-in(x-y)}f(y)\mathrm{d}y$$
$$= \begin{cases}(2N+1)\int_{-\pi}^{\pi} f(y)\mathrm{d}y & x=y\\ \int_{-\pi}^{\pi}\frac{\sin(N+1/2)(x-y)}{\sin 1/2(x-y)}f(y)\mathrm{d}y & x\ne y\end{cases}$$

记$D_N(x) = \dfrac{\sin[(N+1/2)x]}{\sin 1/2x}$，其中$2k\pi$处延拓成$2N+1$，称为Dirichlet核。

那么$S_N f = D_N * f$

有性质（时域卷积=频域乘积）$\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$

证：

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\mathrm{d}y\, e^{-inx}\mathrm{d}x$$
$$\overset{Fubini}{=\!=\!=} \int_{-\pi}^{\pi} g(y)e^{-iny}\cdot\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)e^{-in(x-y)}\mathrm{d}x\Big]\mathrm{d}y = \hat{f}(n)\hat{g}(n)$$

10.2 Good Kernels - Féjer Theorem.

Given a series of kernel function ($2\pi-$periodic) satisfies the following:

a. $\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(x)\mathrm{d}x = 1$; b. $\exists M, \forall n, \int_{-\pi}^{\pi}|K_n(x)|\mathrm{d}x \le M$ (L^1一致有界); c. $\int_{\delta<|x|<\pi}|K_n(x)|\mathrm{d}x \to 0 (n\to+\infty)$

Prop.(Good Kernel Approximation) Given good kernels $\{K_n\}$, f integrable, then if f continuous at x_0, $(K_n * f)(x_0) \to f(x_0)$. Moreover if $f \in C[-\pi,\pi]$, then $(K_n * f)(x) \rightrightarrows f(x)$

Proof:

$$|K_n * f(x_0) - f(x_0)| = \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(y)\big[f(x_0-y)-f(x_0)\big]\mathrm{d}y\Big|$$
$$\le \frac{1}{2\pi}\int_{\delta\le|y|\le\pi}|K_n(y)||f(x_0-y)-f(x_0)|\mathrm{d}y + \frac{1}{2\pi}\int_{|y|\le\delta}|K_n(y)||f(x_0-y)-f(x_0)|\mathrm{d}y$$

第一项$\leq \dfrac{1}{2\pi}\sup|f|\displaystyle\int_{\delta\leq|y|\leq\pi}|K_n(y)|\mathrm{d}y\longrightarrow 0$

第二项：选取适当的δ, $|f(x_0-y)-f(x_0)|<\varepsilon$对$|y|\leq\delta$成立。

从而第二项$\leq \varepsilon M/(2\pi)\longrightarrow 0$

Thus $(K_n*f)(x_0)\to f(x_0)$.

Moreover if $f\in C[-\pi,\pi]$, f is uniformly continuous, thus δ can be choosen uniformly, and we get the result $K_n*f\rightrightarrows f$

Unfortunately Dirichlet Kernel is not a good kernel. $\displaystyle\int_{-\pi}^{\pi}|K_n(x)|\mathrm{d}x\geq c\log n$. 于是我们不能利用Good Kernel逼近给出Fouriér级数的点态收敛性。

然而我们发现Cesaro和具有Good Kernel性质！即取$F_N*f=\dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1}(D_n*f)$，

$F_N=\dfrac{1}{N}\sum D_n$（称为Féjer核）

计算知$F_N=\dfrac{1-\cos Nx}{N(1-\cos x)}$满足Good Kernel。于是这就说明Fouriér级数的Cesaro和具有点态收敛性质。

10.3 L^2逼近

定义$<f,g>=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}f(t)\overline{g}(t)\mathrm{d}t$，其中$f,g:2\pi$-periodic,$[-\pi,\pi]\to\mathbb{C}$，$||f||_{L^2}:=<f,f>^{1/2}$.

下面看到Fourier级数在L^2下逼近效果最好。

（L^2最佳逼近）取$g=\sum_{|n|\leq N}c_ne^{inx}$, c_n 任意选取。

f可积则$||f-S_Nf||_{L^2}\leq||f-g||_{L^2}$

证：$||f-g||_{L^2}^2=||f-S_Nf||^2+||S_Nf-g||^2+2Re<f-S_Nf,S_Nf-g>$

$< f-S_Nf, S_Nf-g >= 0$，于是得证。

做为推论得到：

（L^2收敛）f可积，$||f-S_Nf||_{L^2} \stackrel{N\to+\infty}{\longrightarrow} 0$

（**Parseval**恒等式）$||f||_{L^2} = \sum|\hat{f}(n)|^2$

证：(L^2)：

如果$f\in C[-\pi,\pi]$，$||f-S_Nf||_{L^2} \le ||f-F_N*f||_{L^2} \to 0(N\to+\infty)$

对于一般的f，可以找到一族f_k，在L^1范数下逼近f，并且$|f_k|\le\sup|f|$.(10.0)

由于最佳逼近$||f-S_Nf||_{L^2} \le ||f-S_Nf_k||_{L^2}$

则$||f-S_Nf||_{L^2} \le ||f-f_k||_{L^2} + ||f_k-S_Nf_k||_{L^2}$

第一项$\le \sup|f-f_k|\cdot||f-f_k||_{L^1}$

第二项由前述趋于0.

Parseval显然。

（**Riemann-Lebesgue**定理）$\hat{f}(N)\to 0$

10.4 点态收敛

12.1 Applications of Fourier series

Weyl Equidistribution Thm.

Given an irrational number γ, $\xi_n =< n\gamma >:= n\gamma - [n\gamma]$. It is reasonable to guess that ξ_n distribute equally over the interval $[0,1)$.

This guess is somehow true, and the result is usually called Weyl Equidistribution Theorem.

Rmk. $(\xi_n =< n\gamma >)$ is dense in $[0,1)$ (Using Kronecker's trick)

$\mathscr{Def}.$ Equidistribution. （渐进均匀分布）

A series $\xi_n \in [0,1)$ is called equi-distributed iff $\forall a.b \in [0,1)$,
$$\lim_{N\to+\infty} \frac{\#\{\xi_n | 1 \le n \le N : \xi_n \in [a,b)\}}{N} = b - a$$

The main result is the previously mentioned theorem.

Theorem. (Weyl Equidistribution Theorem) ξ_n is equi-distributed $\iff$ γ is irrational.

To prove the theorem, we claim the following criterion.

Weyl Equidistribution Criterion. An arbitrary series ξ_n is equi-distributed
$$\iff \forall k \in \mathbb{N}\backslash\{0\}, \quad \lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0$$

Assuming the criterion, we apply the criterion to the series $< n\gamma >$, here
$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k n\gamma} = \frac{1}{N}\sum_{n=1}^{N} (e^{2\pi i k\gamma})^n = \frac{1}{N} \cdot \frac{e^{2\pi i k\gamma} - e^{2\pi i k\gamma(N+1)}}{1 - e^{2\pi i k\gamma}} = 0$, hence the result.

Proof of the Criterion.

$\Longleftarrow$.

$$\text{Equi-distribution} \iff \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_{[a,b)}(\xi_n) \overset{N\to+\infty}{\longrightarrow} b - a = \int_0^1 \mathbb{1}_{[a,b)}(t)\mathrm{d}t$$

Then it suffices to prove the following statement $\lim\limits_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(t)\mathrm{d}t$ holds when f is a step function.

Step 1. Continuous cases.

Suppose the Weyl's Criteria, $f \in C([0,1])$, and periodic 1. Then the following holds.

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(t)\mathrm{d}t$$

Recalling that $F_N * f \rightrightarrows f$, here F_N is the Fejer Kernel.

Then

$$|\frac{1}{N} \sum_{n=1}^{N} f(\xi_n) - \int_0^1 f(t)\mathrm{d}t| \leq \frac{1}{N} \sum_{n=1}^{N} |f - (F_M * f)|(\xi_n) + |\frac{1}{N} \sum_{n=1}^{N} (F_M * f)(\xi_n) - \int_0^1 (F_M * .$$
$$+ \int_0^1 |F_M * f - f|(t)\mathrm{d}t$$

2nd term holds since $F_M * f$ has the form $\sum a_m e^{2\pi ikt}$, through direct check one can verify it can be controlled by increasing N, after choosing a satsfying n, 1st and 3rd term can be controlled by adjusting M.

Thus we prove the continuous cases.

Step 2. Step Function cases.

For an arbitrary step function (or more specificly $\mathbb{1}_{[a,b]}$), we have the following continuous function approximation.

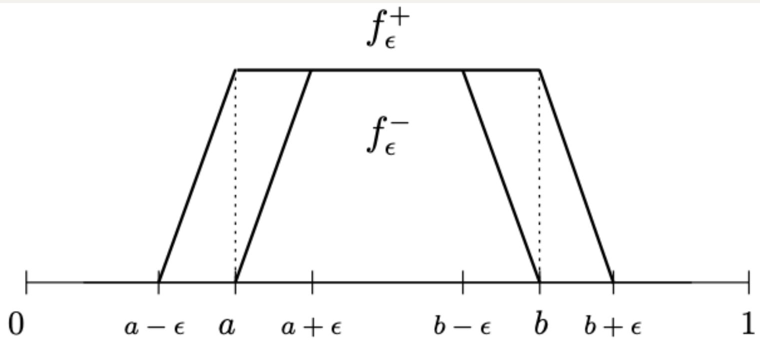

Figure 3. Approximations of $\chi_{(a,b)}(x)$

Obv. that $f_\varepsilon^+ \geq f \geq f_\varepsilon^-$

By taking limits (over ε), we obtain that for an arbitrary step function f.

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}f(\xi_n)=\int_0^1 f(t)\mathrm{d}t$$

($\limsup\sum f\leq\limsup\sum f^+=\int f^+=\int f+\varepsilon$, similarly we get the other side)

Since linear relation, indicator function cases holds implies the entire step function cases.

Thus we get the $\Longleftarrow$ direction.

Remark. A useless step 3. Riemann Integrable Function cases.

$\lim\limits_{N\to+\infty}\frac{1}{N}\sum\limits_{n=1}^{N}f(\xi_n)=\int_0^1 f(t)\mathrm{d}t$ also holds for f Riemann integrable over $[0,1]$

By approaching f using step function from both sides, we get desired results.

$\Longrightarrow$.

$\frac{1}{N}\sum\limits_{n=1}^{N}\mathbb{1}_{[a,b)}(\xi_n)\xrightarrow{N\to+\infty}b-a=\int_0^1\mathbb{1}_{[a,b)}(t)\mathrm{d}t$ holds. Thus
$\lim\limits_{N\to+\infty}\frac{1}{N}\sum\limits_{n=1}^{N}f(\xi_n)=\int_0^1 f(t)\mathrm{d}t$ holds for Step functions, thus Riemann Integrable functions, thus Continuous Function. Namely $\lim\limits_{N\to+\infty}\frac{1}{N}\sum\limits_{n=1}^{N}f(\xi_n)=\int_0^1 f(t)\mathrm{d}t$ holds for $e^{2k\pi ix}\iff Continuous\iff Step\iff Riemann\ Integrable$

Thus we get the result.

Heat Equation on a circle

Question: $\begin{cases}\partial_t u=\partial_{xx}u\\ u(0,x)=f(x)\end{cases}$, u C^1 over t, C^2 over x, periodic on x.
Solution: **Step 1. Separation of variable: Suppose** $u(t,x)=A(t)B(x)$**, with** $A,B\neq 0$
Thus $\frac{A'(t)}{A(t)}=\frac{B''(x)}{B(x)}$

Adjusting x, t, we have $\dfrac{A'(t)}{A(t)} = \dfrac{B''(x)}{B(x)} = \lambda$.

Thus $B(x) = e^{2\pi inx}$, since periodic. Then we get $\lambda = -4\pi^2 n^2$, $A(t) = e^{-4\pi^2 n^2 t}$.

Step 2. Linear operations.

We get a series of solutions $u(t, x) \sim \sum_n c_n e^{-4\pi^2 n^2 t} e^{2\pi inx}$

To obtain the boundary condition, it is easy to verify that $c_n = \hat{f}(n)$.

16.2 Fourier Transform

13. Measure Theory

Following Contents are Real Analysis Parts.

13.1 Measure

Definition. ($\sigma-$algebra) $\mathcal{F} \subseteq P(E)$ is a $\sigma-$algebra over the set E if it satisfies the following 3 conditions.

1. $E \in \mathcal{F}$

2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$

3. $A_{i \in I} \in \mathcal{F} \implies \cup_{i=1}^{+\infty} A_i \in \mathcal{F}$, I a countable index set.

($\sigma - Closure$) Given a $C \subseteq P(E)$, $\sigma(C) = \cap_{\mathcal{F} \supseteq C} \mathcal{F}$

Given arbitarily many $\sigma-$algebras $\mathcal{F}_n$, $\cap \mathcal{F}_n$ is also a $\sigma-$ algebra.

(Borel Sets) For an Topological Space X, its open sets are O, $B(E) = \sigma(O)$

Definition. (Measure) Given $(E, \mathcal{F})$, a function $\mu : \mathcal{F} \to [0, +\infty]$ is called a measure if it satisfies:

1. $\mu(\varnothing) = 0$, **2.** $\mu(\sqcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n)$ **(σ-addictivity)**

Examples. 1. $E = \mathbb{Z}, \mathcal{F} = P(A), \mu(A) = |A|$;

Caratheodory Construction.

$\underset{pre\ measure}{\mu:\mathcal{S}\to[0,\infty]} \longrightarrow \underset{outer\ measure}{\mu^*:2^X\to[0,\infty]} \xrightarrow{Caratheodory\ Condition} \underset{measure}{\bar{\mu}:\mathcal{M}=\mu^*-measurable\to[0,\infty]}$

Take the semi-ring $\mathcal{S} = bounded\ intervals, \mu\ map\ interval\ to\ its\ length.$

Through this construction (and using some conclusions of Caratheodory-Hahn Thm.) we get a measure $\bar{\mu}$ (Actually a unique expension on $\mathcal{M}$) on $\mathcal{M}$, where $\mathcal{M}$ is a σ algebra containing intervals. Thus $\mathcal{M}$ contains Borel sets.

Prop. μ **a measure over** $(E, \mathcal{F})$

1. $A \subseteq B \implies \mu(A) \leq \mu(B)$

2. $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$

3. $A_n \subseteq A_{n+1}$**,** $\mu(\cup_{n=1}^{+\infty} A_n) = \lim_{n\to+\infty} \mu(A_n)$

4. $B_n \supseteq B_{n+1}$**,** $\mu(B_1) < +\infty$**,** $\mu(\cap_{n=1}^{+\infty} B_n) = \lim_{n\to+\infty} \mu(B_n)$

5. $\mu(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$

13.2 Integration

$f(x) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}, A_i \in \mathcal{F}$, disjoint.

Define $\displaystyle\int_E f \mathrm{d}\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i)$

Definition (Measurable Function) $f : (E_1, \mathcal{F}_1) \to (E_2, \mathcal{F}_2)$ **is measurable if** $\forall A \in \mathcal{F}_2, f^{-1}(A) \in \mathcal{F}_1$

If $(E_2, \mathcal{F}_2) = (\mathbb{R}, Borel)$, we usally call f a $\mathcal{F}_1$ measurable funtion.

Definition (Integration)

1. $f \geq 0, \int_E f\mathrm{d}\mu = \sup_{h\leq f, h\ simple} \int_E h\mathrm{d}\mu$

2. $\int_E f\mathrm{d}\mu = \int_E f^+\mathrm{d}\mu - \int_E f^-\mathrm{d}\mu$

Remark. Say f integrable if $\displaystyle\int |f|\mathrm{d}\mu < +\infty$, then $\displaystyle\int_E f^+, \int_E f^- < +\infty$

Rimeann Integrable On Closed Interval $\implies$ **Measurable and** $\int_L = \int_R$

Proof: $\int_L = \int_R$ if we use step functions as the simple functions. Using Upper sums and Lower sums immediately leads to the result.

For the measurable part,

14. Probability

We use Kolomogrov axiom to define the probability.

A probability space consists of $(\Omega, \mathscr{F}, \mathbb{P})$, where Ω is the hole space, $\mathscr{F}$ is a $\sigma-$algebra, and $\mathbb{P}$ a measure over it such that $\mathbb{P}(\Omega) = 1$

(Independence of events) $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2)$**, then** E_1, E_2 **independent.**

More generally say a countable family of events $\{E_i\}_{i=1}^{+\infty}$ **independent if** $\mathbb{P}(\cap E_i) = \prod \mathbb{P}(E_i)$ **holds for any sub family** $E_i (i \in I)$

(Condition Probability) For $A \in \mathscr{F}, \mathbb{P}(A) > 0$**, then for** $E \in \mathscr{F}$**, define the condition probability** $\mathbb{P}(E|A) = \dfrac{\mathbb{P}(E \cap A)}{\mathbb{P}(A)}$

Through standard checking process this indeed defines a probability on $\mathscr{F}$

Similar to sets, we have Product space by conducting cartesian product.

(Random Variables) A random variable is a measurable function $X : \Omega \to \mathbb{P}$

(Discrete Random Variable) $\mathbb{P}(X = X_i) = P_i$, X ranges on discrete countable points.

(Independence of Random Variables) Given a family of random variables $(X_i)_{i \in I}$, they are independent if $\forall \alpha_1, \cdots, \alpha_n \in I$, $\forall F_{\alpha_1}, \cdots, F_{\alpha_n} \in B(\mathbb{R})$, we have $\mathbb{P}(\cap\{X_{\alpha_i} \in F_{\alpha_i}\}) = \prod \mathbb{P}(\{X_{\alpha_i} \in F_{\alpha_i}\})$

(Expectation) $E(X) = \int X \mathrm{d}\mathbb{P}$

(Variance) $Var(X) = E((X - E(X))^2) = E(X^2) - E(X)^2$

Prop. If $X_1, \cdots, X_n$ are independent random variables, then $Var(\sum X_i) = \sum Var(X_i)$

Proof:
$LHS = E[(\sum x_i)^2] - (\sum E[x_i])^2 = \sum(E[x_i^2] - (E[x_i])^2) + \sum(E[x_i x_j] - E[x_i]E[x_j]) = RH$

Examples for discrete random variables.

Bernoulli $X \in \{0, 1\}, P(X = 1) = p; P(X = 0) = 1 - p$

Binomial

$\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

$\sum^n Bernoulli(p) \sim Binomial(n, p), E(Binomial(n, p)) = nE(Bernoulli(p)) = np$

Geometry

$\mathbb{P}(X = k) = (1 - p)^{k-1} p, E(X) = 1/p$

Poisson

$\mathbb{P}(X=k)=e^{-\lambda}\cdot\frac{\lambda^k}{k!}, E(X)=Var(X)=\lambda$

Markov Inequality $\mathbb{P}(|x|>\lambda)\leq\frac{1}{\lambda}E(|x|)$

This is exactly same to the Chebyshev inequality in the real analysis case.

Weak LLN

X_i i.i.d.$\to X, \sum X_n/N \to (under\ probability) E[X]$

Proof:
$$\mathbb{P}[|S_n-E[X]|\geq\varepsilon]=\mathbb{P}[(S_n-E[S_n])^2\geq\varepsilon^2]$$
$$\leq E[(S_n-E[S_n])^2]/(\varepsilon^2)=(iid)=\sum Var[X_n]/n^2\leq C/n$$

15. Random Graph & Perrolation

Graph are defined to be 1-skeleton of a CW complex.

15.1 Erdos-Renyi Graph

Model $G=G(n,P_n)$, with n vertices. Sample i.i.d. Bernoulli random variables of parameter P_n. Connect i,j if $X_{i,j}=1$, otherwise 0.

Connectivity

$P_n=1$, then $\mathbb{P}(G=K_n)=1$.

Connectivity Theorem.

1. For $P_n\geq\frac{(1+\varepsilon)\log n}{n}, \varepsilon>0$**, then** $\mathbb{P}(G(n,p)\ is\ connected)\xrightarrow{n\to+\infty}1$

2. For $P_n\leq\frac{(1-\varepsilon)\log n}{n}, \varepsilon>0$**, then** $\mathbb{P}(G(n,p)\ is\ connected)\xrightarrow{n\to+\infty}0$

We call case 1 "supercritical"(超临界), case 2 "subcritical"(次临界)

In this case $\frac{\log n}{n}$ here is the critical situation.

Proof:

Lemma 1. (Isolated point)

For $P_n \geq \dfrac{(1+\varepsilon)\log n}{n}$**,** $\mathbb{P}(G(n,p)\ has\ isolated\ point) \xrightarrow{n\to+\infty} 0$

Lemma 1 Proof:

First Moment Method (Mean/Expectation): 一阶矩方法

$\mathbb{P}(\#isolated\ vertices\ of\ G(n,p) \geq 1) \leq \mathbb{E}(\#isolated\ vertices)$

To compute the last term, we observe that $\#isolated\ vertices = \sum_{i=1}^{n} \mathbb{1}_{i\ is\ isolated}$

Thus $\mathbb{E}(\#isolated\ vertices) = \sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{i\ is\ isolated}) = \sum_{i=1}^{n} \mathbb{P}(i\ is\ isolated)$

Now For any single point i, $\mathbb{P}(i\ is\ isolated) = \mathbb{P}(X_{i,t} = 0 \quad \forall t \neq i) = (1-P_n)^{n-1}$

Hence:

$$\mathbb{E}(\#isolated\ vertices) = n(1-P_n)^{n-1} \leq n(1-\frac{(1+\varepsilon)\log n}{n})^{n-1}$$
$$\leq n(e^{-\frac{(1+\varepsilon)\log n}{n}})^{n-1} \leq n^{1-(1+\varepsilon)(n-1)/n} \xrightarrow{n\to+\infty} 0$$

We get a conclusion $n(1-P_n)^n \simeq n^{-\varepsilon}$.

Lemma 2. (Isolated point)

For $P_n \leq \dfrac{(1+\varepsilon)\log n}{n}$**,** $\mathbb{P}(G(n,p)\ has\ isolated\ point) \xrightarrow{n\to+\infty} 1$

Lemma 2 Proof:

Second Moment Method (Variance): 二阶矩方法

Denote $X = \#\{isolated\ vertices\}$, then $\mathbb{P}(X \geq 1) \geq \dfrac{\mathbb{E}^2(X)}{\mathbb{E}(X^2)}$(Cauchy-Schwartz)

Since $\mathbb{E}(X^2) = \mathbb{E}\left((\sum \mathbb{1}_{i\ is\ isolated})^2\right) = \sum \mathbb{E}(\mathbb{1}_{i\ is\ iolated}) + \sum \mathbb{E}(\mathbb{1}_{i,j\ both\ are\ isolated})$

The first sum is $n(1-P_n)^{n-1}$, second sum is $n(n-1)(1-P_n)^{2n-3}$(through elementary combinatorics)

Thus $\dfrac{\mathbb{E}^2(X)}{\mathbb{E}(X^2)} = [n^2(1-P_n)^{2n-2}]/[n(n-1)(1-P_n)^{2n-3} + n(1-P_n)^{n-1}] \to 0$

Main Proof of Part 1.

$$\mathbb{P}(G\ isn't\ connected) = \mathbb{P}(\exists cut\ V_1 \sqcup V_2)$$
$$\leq \mathbb{P} \cup_{k=1}^{[n/2]} (\exists cut, |V_1| = k,\ no\ edges\ between\ V_1, V - V_1)$$
$$\leq \sum_{k=1}^{[n/2]} \binom{n}{k} (1-P_n)^{k(n-k)}$$

Since P_n is monotonic w.r.t. ε, it is possible to let ε be arbitarily small.

Condition: $n >> 1$.

Since
$$\binom{n}{k}(1-P_n)^{k(n-k)} = \frac{n!}{(n-k)!}(1-P_n)^{kn}\frac{1}{k!}(1-P_n)^{-k^2} \simeq n^k \cdot (1-P_n)^{nk}(k/e)^{-k}e^{P_n k^2}$$
$$\simeq n^{-\varepsilon k}(k/e)^{-k}e^{(1+\varepsilon)k^2 \log n/n}$$
$$\leq \exp(-\varepsilon k \log n - k\log(k/e) + (1+\varepsilon)k^2 \log n/n)$$

$= \exp(-k[\varepsilon \log n + \log(k/e) - (1+\varepsilon)(k/n)\log n])$

If $k \leq \dfrac{\varepsilon}{2}n, \varepsilon \log n - (1+\varepsilon)(k/n)\log n \geq \varepsilon/3 \cdot \log n$

If $n/2 \geq k \geq \varepsilon n/2$,
$\varepsilon \log n + \log(k/e) - (1+\varepsilon)(k/n)\log n \geq \log((\varepsilon n)/(2e)) - (1+\varepsilon)/2 \cdot \log n$
$$= (1-\varepsilon)/2 \cdot \log n + \log[\varepsilon/(2e)] > (\log n)/3$$

Thus
$$\mathbb{P}(G\ isn't\ connected) \leq \sum e^{-k[\varepsilon \log n + \log(k/e) - (1+\varepsilon)(k/n)\log n]}$$
$$\leq \sum_{k=1}^{+\infty} n^{-\varepsilon k/3} + \sum_{k=[\varepsilon n/2]}^{+\infty} n^{-k/3} = n^{-\varepsilon/3}/(1-n^{-\varepsilon/3}) + n^{-\varepsilon n/6}/(1-n^{-1/3}) \xrightarrow{n\to+\infty} 0$$

Proof of Part 2.

Directlty deduced from Lemma 2.

15.2 Bernoulli Percolation

Definition. (Bernoulli Percolation: 伯努利渗流模型) Given a graph $G = (V, E)$, associate every $e \in E$ with an independent Bernoulli Random variable $\omega(e)$ with the parameter p. (This is called the distribution of a percolation configuration)

Given the percolation configuration distribution, we define a new random graph $G = (V, \mathcal{E})$, where for every e, $e \in \mathcal{E} \iff \omega(e) = 1$, call the edge open, otherwise closed.

Erdos-Renyi graph, particularly, is the case when taking $G = K_n$

Remarks: Here the probability space is
$(\{0,1\}^E, \mathcal{F} = Events\ generated\ by\ Finite\ Edges\ Events, \mathbb{P}_p)$

Now we consider the Bernoulli Percolation on $\mathbb{Z}^d$, i.e. take $G = (\mathbb{Z}^d, E(\mathbb{Z}^d))$, where $E(\mathbb{Z}^d) = \{(x,y)|dist(x,y) = 1\}$.

$\theta(p) = \mathbb{P}(0 \longleftrightarrow \infty) = \mathbb{P}(\{\exists an\ infinite\ component\ containing\ 0\})$, where $(x \longleftrightarrow y)$ denote x, y can be connected by a open path.

Here $(0 \longleftrightarrow \infty) = \cap_{n\geq 1}(0 \longleftrightarrow \partial\Lambda_n)$, where Λ_n is the square $[-n, n]^d$, hence a finite edge event, hence $(0 \longleftrightarrow \infty)$ indeed exists in the event domain.

Call the component containg 0 $C(0)$ the cluster containing 0.

We would like to consider the critical probability in which $\theta(p)$ changes from 0 to non-0 values.

Prop 1.(Lower case) $\forall d \in \mathbb{N}_+, \exists \tilde{p}(d)$, **such that** $p \leq \tilde{p}(d) \implies \theta(p) = 0$

Proof: Since $(0 \longleftrightarrow \infty) = \cap_{n\geq 1}(0 \longleftrightarrow \partial\Lambda_n)$, it is also a descending chain, thus $\mathbb{P}(0 \longleftrightarrow \infty) = \lim_{n\to\infty}(0 \longleftrightarrow \partial\Lambda_n)$

For each n,

$$\mathbb{P}(0 \longleftrightarrow \partial\Lambda_n) \leq \sum_{k\geq n} \mathbb{P}(\exists length\ k - open\ path : 0 \longleftrightarrow \partial\Lambda_n)$$
$$\leq \sum_{k\geq n} p^k \cdot (\#length\ k - open\ path : 0 \longleftrightarrow \partial\Lambda_n)$$
$$\leq \sum_{k\geq n} p^k \cdot (2d)^k = (2dp)^n \cdot \frac{1}{1-2dp}$$

Take $p < 1/(2d)$, we get the result. Thus $\widetilde{p}(d) = 1/(2d) - \varepsilon$ is a suitable choice.

Prop 2.(Upper case) $\forall d \geq 2, \exists \tilde{\tilde{p}}(d) \in (0,1), p \geq \tilde{\tilde{p}}(d) \implies \theta(p) > 0$

Proof: Similarly we have
$\mathbb{P}(0 \longleftrightarrow \infty) = 1 - \mathbb{P}(0 \not\longleftrightarrow \infty) = 1 - \lim_{n\to\infty} \mathbb{P}(0 \not\longleftrightarrow \partial\Lambda_n)$

It suffices to prove that for $p > \tilde{\tilde{p}}, \mathbb{P}(0 \not\longleftrightarrow \partial\Lambda_n) \leq c \in (0,1)$

$$\mathbb{P}(0 \not\longleftrightarrow \partial\Lambda_n) = \mathbb{P}(\exists a\ open\ dual\ hyperplane\ separating\ 0\ and\ \partial\Lambda_n)$$
$$\leq \sum_{k\geq 2d} \mathbb{P}(\exists a\ size - k\ hyperplane\ separating\ 0\ and\ \partial\Lambda_n)$$
$$\leq \sum_{k\geq 2d} c(d)^k(1-p)^k = \frac{[c(d)(1-p)]^{2d}}{1-[c(d)(1-p)]}$$

Take a rather large p, (near to 1), we can let $\mathbb{P}(0 \not\longleftrightarrow \partial\Lambda_n)$ be controlled by a rather small value.

Prop 3.(Monotonicity) $\theta(p)$ **is increasing.**

Proof: Define a event A is increasing, if $\omega \in A, \omega' \geq \omega$, then $\omega' \in A$. Where $\omega, \omega' \in \{0,1\}^E, \omega' \geq \omega$ if for all components we have the same greater relation.

We claim if A is increasing, $p < p', \mathbb{P}_p(A) \leq \mathbb{P}_p(A')$

Proof:

Coupling Method:耦合方法.

Given 2 probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$

Construct a new probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \tilde{\mathbb{P}})$, such that

$$\tilde{\mathbb{P}}(A_1 \times \Omega_2) = \mathbb{P}_1(A_1)$$

$\tilde{\mathbb{P}}(\Omega_1 \times A_2) = \mathbb{P}_2(A_2)$.

Note that product space is a coupling space.

Stochastic domination

Given 2 real value random variable $X_1 \in (\Omega_1, \mathcal{F}_1, \mathbb{P}_1), X_2 \in (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$

We construct a coupling space such that $\tilde{\mathbb{P}}(X_1 \leq X_2) = 1$.

For example, consider 2 coin toss with front face probability p_1, p_2, resp.

We sample a uniform distributed random variables $U \overset{distribution}{\sim} Unif([0,1])$

Let $X_1 = \mathbb{1}_{U \leq P_1}, X_2 = \mathbb{1}_{U \leq P_2}$, then $X_1 \leq X_2$.

Back to the proposition:

Take i.i.d. random variables which distributed uniformly on $(0, 1)$. $\{U_e, e \in E(\mathbb{Z}^d)\}$

Take $\omega_p(e) = \mathbb{1}_{\{U_e \leq p\}}$,

Then clearly $\omega_p \sim \mathbb{P}_p, \omega_p \leq \omega_{p'}$ if $p < p'$. Thus
$\mathbb{P}_p[A] = \tilde{\mathbb{P}}[\omega_p \in A] \leq \tilde{\mathbb{P}}[\omega_p' \in A] = \mathbb{P}_{p'}[A]$

Theorem. (Phase Transition) $d \geq 2, \exists p_c(d) \in (0,1)$, **such that**
$p < p_c \implies \theta(p) = 0;$
$p > p_c \implies \theta(p) = 1$.

More particularly, when $d = 2, or\ d \geq 11, \theta(p_c) = 0$; when $3 \leq d \leq 10$, determine $\theta(p_c)$ value is a Fields-Medal-Level Work.

Lem. $p \to \theta(p)$ **is right-continuous.**

Thm. Kolmogrov 0-1 law

16. Markov Chain

Definition. (Markov Chain) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with a series of random variables $(X_n)_{n\in\mathbb{N}}$.

We call the series a Markov Chain if
$\mathbb{P}(X_{n+1} \in E_{n+1} | \cap_{1\leq i\leq n} (X_i \in E_i)) = \mathbb{P}(X_{n+1} \in E_{n+1} | X_n \in E_n)$ **holds for all** $E_i \in Borel(\mathbb{R})$**. (i.e. only the present state matters)**

E.g. If
$\mathbb{P}(X_{n+1} \in E_{n+1} | \cap_{1\leq i\leq n} (X_i \in E_i)) = \mathbb{P}(X_{n+1} \in E_{n+1} | (X_n \in E_n) \cap (X_{n-1} \in E_{n-1}))$

Then $Y_n = (X_n, X_{n-1})$ $(\Omega^2 \to \mathbb{R}^2)$ is a $\mathbb{R}^2-$valued random variable.

Hence $\mathbb{P}(Y_{n+1} \in E_{n+1} \times E_n | Y_i) = \mathbb{P}(Y_{n+1} \in E_{n+1} \times E_n | Y_n \in E_n \times E_{n-1})$, and indeed the sigma closure of $Borel(\mathbb{R})^2$ is $Borel(\mathbb{R}^2)$, thus a markov chain.

Definition. (Homogeneous Markov Chain; Transition Matrix) A countable state markov series is times homogeneous if $\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n) = Q_{x_n, x_{n+1}}$

Here the matrix Q is a fixed matrix.

Remark. $\forall x, \sum_y Q_{x,y} = 1$

Prop. (X_n) **times homogeneous Markov Chain with transition matrix** Q**,** $X_0 \overset{d}{\sim} Q$**, then** Q, μ **determine the distribution of the markov chain** X_n

Proof: By induction and using the transition matrix.

$\mathbb{P}(X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n)$

$$= \mathbb{P}(X_n = x_n | X_0 = x_0 \cap \cdots \cap X_{n-1} = x_{n-1}) \cdot \mathbb{P}(X_0 = x_0, \cdots, X_{n-1} = x_{n-1})$$
$$= Q_{x_{n-1}, x_n} \cdot \mathbb{P}(X_0 = x_0, \cdots, X_{n-1} = x_{n-1})$$
$$= \cdots = \mu_{x_0} \cdot Q_{x_0, x_1} \cdots Q_{x_{n-1}, x_n}$$

Corollary. $\mathbb{P}(X_n = x_n) = (\mu Q^n)_{x_n}$

Simple Random Walk on Graph

Definition. (SRW on Graph) $G = (V, E), \deg V < +\infty$**, a SRW on** G **is a markov chain s.t.** $\mathbb{P}(X_{n+1} = \nu | X_n = \mu) = \dfrac{1}{\deg(\mu)} \mathbb{1}_{\mu\sim\nu}$

Note $\mathbb{P}_x$ denotes the markov chain starting from point x.

Prop. $f_n(x) = \mathbb{E}_x[f(X_n)]$, then $f_n(x) = (Q^n f)(x)$. $f : V \to \mathbb{R}$.

Prop. $f_{n+1} - f_n = Lf_n$, where L is defined to be $(Q - I)$.

$Lf_n = \sum_{y, y\sim x} Q_{x,y}(f(y) - f(x))$

Remark. Note the astonishing relation between the previous proposition and the discrete heat equation, we can establish an amazing method to solve the heat equation numerically.