Galois descent in algebraic K-theory

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Abstract. This is the outline of the presentation in Prof. Dan Isaksen's weekly online meeting. This presentation is intended to show that localized algebraic K-theory satisfies some certain descent property such as Galois and some general case. The main reference is [CMNN17].

Contents

1 Descent for algebraic K-theory

1.1 Historical remarks

A long-standing problem for algebraic K-theory is that we'd like to examine its descent property. One of probably the most ancient result of this type is Mayer-Vietoris sequence for Milnor square.

Theorem 1.1 (Mayer-Vietoris)**.** *Given a Milnor square,i.e.*

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow[\bar{f}]{} & S/I \end{array}$$

such that I is an ideal of R and I is mapped isomorphically into S. We have the following long exact sequence:

$$\cdots K_{n+1}(S/I) \to K_n(R) \to K_n(S) \oplus K_n(R/I) \to K_n(S/I) \to \cdots$$

Remark. This basically comes from Milnor patching, i.e. we have cartesian square

$$\begin{array}{ccc} \mathbf{Proj}(R) & \longrightarrow & \mathbf{Proj}(S) \\ \downarrow & & \downarrow \\ \mathbf{Proj}(R/I) & \longrightarrow & \mathbf{Proj}(S/I) \end{array}$$

On the other hand, we have topologies for schemes and we naturally wonder how algebraic K-theory behaves under these topologies. The main and classical result is that algebraic K-theory do satisfies Nisnevich descent.[TT90, Section 10]

A natural hope is that algebraic K-theory furthermore satisfies étale descent. This is strongly related with Lichtenbaum-Quillen conjecture which compares algebraic K-theory and étale cohomology.

These type of results basically due to work of Thomason and Voevodsky-Rost:

Definition 1.2. Define mod$p$ K-theory as $K(R;\mathbb{Z}/p\mathbb{Z}) \cong K(R) \otimes \mathbb{S}/p$. Consider the v_1-self map $\beta : \Sigma^{2p-2}\mathbb{S}/p \to \mathbb{S}/p$, we define the periodic algebraic K-theory as the telescope of β, i.e.

$$K(R,\mathbb{Z}/p\mathbb{Z})[\beta^{-1}] \cong K(R) \otimes \mathbb{S}/p[\beta^{-1}] \cong L_{T(1)}K(R) \otimes \mathbb{S}/p$$

(The last equality comes from $L_{T(1)}$ is smashing)

Theorem 1.3 ([TT90, Theorem 11.5])**.** *The periodic K-theory $L_{T(1)}K$ satisfies étale descent. Moreover, under suitable conditions on scheme X, $L_{T(1)}K(X) \simeq K^{\acute{e}t}(X)$.*

Remark. In fact, this reduce Lichtenbaum-Quillen to the upper boundedness of $\mathbb{S}/p \otimes \mathrm{fib}(K \to L_1K)$, which is what we refer to by saying Lichtenbaum-Quillen property in modern text.

The following work given by Voevodsky-Rost establish a precise isomorphism between Milnor-K theory and étale cohomology.

Theorem 1.4 ([Voe11])**.** *Suppose k is a field of characteristic zero which contains a primitive l-th root of unity, then*

$$K_n^M(k)/l \to H^n_{\acute{e}t}(k, \mu_l^{\otimes n})$$

are isomorphisms for all n.

We conclude this subsection by stating the (probably) newest result on étale descent for K-theory which is actually a future work of this paper.

Theorem 1.5 ([CMNN21]). *Let X be a qcqs spectral algebraic space, then the natural map*

$$K^{ét}(X) \to L_{K(1)}K(X) \times_{L_{K(1)}TC(X)} TC(X)$$

is an isomorphism on homotopy in degrees ≥ -1.

1.2 Galois descent

We now turn to the main topic of this presentation. Now that we've already known algebraic K-theory do satisfies Nisnevich descent, it suffices to only check descent property for Galois extensions (by [LurDAG11, Corollary 4.24]: Nisnevich+Galois=Étale, just as classical commutative algebra and scheme theory).

The main result we want to show in this presentation is that: under certain mild conditions, localized K-theory do satisfy Galois descent, hence étale descent (since Bousfield localization preserves pushouts, hence preserve Nisnevich descent property).

Theorem 1.6 (Galois descent). *Let $A \to B$ be a G-Galois extension of $\mathbb{E}_\infty$-ring spectra with G finite. Suppose that the image of the transfer map $K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ contains the unit, then both morphisms*

$$L_n^f K(A) \to L_n^f(K(B)^{hG}) \to (L_n^f K(B))^{hG}$$

are equivalences.

We briefly outline the proof method, which is almost same as the method Thomason used in his proof on étale descent property for $L_1 K$. The main idea is that we consider the $\otimes$-ideal $\mathcal{I}$ consists of M such that

$$M \to M \otimes [B] \rightrightarrows M \otimes [B] \otimes [B] \cdots$$

is indeed a limit diagram. Then we manage to show that $[A] \in \mathcal{I}$,i.e. $R' = \mathrm{Hom}_{Mot_A/\mathcal{I}}(\mathbf{1}, [A])$ is trivial, which can be achieved by requiring some certain mild conditions on the ring map $A \to B$.

Remark. Here we use motive since we only need to show the descent property after taking additive invariant K.

However to take finite localization into account, we can furthermore loosen the condition that the above diagram need not to be an actual limit diagram, but only a limit diagram after the finite localizations. We'll establish ε-enlargement to describe this property.

At the end of section, we'd also like to mention the result doesn't need to be restricted to Galois extensions, since the limit diagram we examine in the main proof is actually a faithful-flat-descent-type diagram. We'll prove an abstract result and then illustrating some explicit examples including Galois descent and some other descent property for ring extensions.

1.3 Notations

1. $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is the ∞-category of idempotent-complete, small, stable ∞-categories and exact functors between them, endowed with Lurie tensor product.

2. Define the commutative algebra objects in it to be $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$.

   Given $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$, let $\mathbf{1}$ be its unit, given an object $X \in \mathcal{C}$, write $\pi_k(X) = \pi_k \operatorname{Hom}_{\mathcal{C}}(\mathbf{1}, X)$.

3. Pr^L_{st} is the ∞-category of presentable stable ∞-categories and cocontinuous functors between them, endowed with Lurie tensor product. Similarly define Pr^L.

4. We use $\widehat{\mathrm{Cat}_\infty^{\mathrm{perf}}}$ to represent the ∞-category of not necessarily small, idempotent-complete stable ∞-categories and exact functors between them. We ignore the slight set-theoretic problem and still freely talk about $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$.

5. L_n denote the Bousfield localization at Morava E theory, i.e. L_{E_n}. L_n^f denote the finite localization of type n. By saying finite localization, we refer to the localization $L_{T(0)\oplus \cdots T(n)}$ where $T(i)$ is a telescope of a finite complex of type n.

   We also remark that these localizations can be upgraded as a localization in presentable stable ∞-categories, since these are modules over Sp.

6. A thick subcategory $\mathcal{T}$ of a stable, idempotent-complete ∞-category $\mathcal{C}$ is a stable full subcategory which is also idempotent-complete.

Remark. It is equivalent to say that a thick subcategory of a stable, idempotent-complete ∞-category is a full subcategory such that it contains zero object, closed under cofiber, fiber and retract.

This is because by [HA, Lemma 1.2.4.6], the problem can be reduced to homotopy category (which is triangulated). Now one finds that closed under retracts (in homotopy category) is precisely closed under direct summands, however given an idempotent $e : A \to A$, since $\mathcal{C}$ idempotent complete, we have split idempotent in $\mathcal{C}$, say $A \to B \to A$. One directly find that $B \to A \to A/B$ left split, hence $A \simeq B \oplus A/B$ in the homotopy category, hence B is also in this subcategory, which shows the equivalences.

2 ε-nilpotence: deviation in the sense of finite localizations

2.1 ε-enlargement

Definition 2.1 (ε-enlargement)**.** Let $\mathcal{C} \in \widehat{\mathrm{Cat}_\infty^{\mathrm{perf}}}$ and $\mathcal{T} \subseteq \mathcal{C}$ a thick subcategory.

1. Given a finite spectrum F, define $\mathcal{T}_F$ to be the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{T}$ and $\{F \otimes C\}_{C \in \mathcal{C}}$

2. Let Σ be a finite set of prime numbers, define thick subcategory

$$\mathcal{T}_{\varepsilon,\Sigma} = \bigcap_F \mathcal{T}_F$$

where F ranges over all finite spectra whose p-localization is nontrivial for every $p \in \Sigma$

3. Define the ε-enlargement of $\mathcal{T}$ to be:

$$\mathcal{T}_\varepsilon = \bigcup_\Sigma \mathcal{T}_{\varepsilon,\Sigma}$$

We also have the "absolute" version of ε-enlargement.

Definition 2.2 (ε-object)**.** Suppose $\mathcal{T} = \{0\}$, we write $Nil_\varepsilon(\mathcal{C}) = \mathcal{T}_\varepsilon$ the subcategory of ε-objects. Similarly we write $Nil_{\varepsilon,\Sigma}(\mathcal{C})$.

A morphism $f : X \to Y$ is called an ε-equivalence if its cofiber is an ε-object.

We only demonstrate some crucial property of ε-enlargement.

By definition it is preserved under exact functor:

Proposition 2.3. *$G : \mathcal{C} \to \mathcal{D}$ in $\widehat{\mathrm{Cat}_\infty^{perf}}$ (an exact functor), suppose $\mathcal{T} \subseteq \mathcal{C}, \mathcal{T}' \subseteq \mathcal{D}$ are thick subcategories, $G(\mathcal{T}) \subseteq \mathcal{T}'$, then $G(\mathcal{T}_\varepsilon) \subseteq \mathcal{T}'_\varepsilon$*

Proof. We only show that $G(\mathcal{T}_F) \subseteq \mathcal{T}'_F$, which can be reduced to the problem that G commutes with smashing with F, i.e. $F \otimes G(X) = G(F \otimes X)$.

This is because G is exact, hence $G((F \cup \mathbb{S}[n]) \otimes X) = G(F \otimes X \cup \mathbb{S}[n] \otimes X) = G(F \otimes X) \cup G(\mathbb{S}[n] \otimes X)$.

We proceed induction on cell structure on F, hence by induction hypothesis, $G((F \cup \mathbb{S}[n]) \otimes X) = X \otimes G(F) \cup X[n] = (F \cup \mathbb{S}[n]) \otimes X$. □

The key characterization of ε-enlargement is that it vanishes under finite localizations.

Proposition 2.4. *Let $\mathcal{C} \in \widehat{\mathrm{Cat}_\infty^{perf}}$ and let $\mathcal{T}$ be a thick subcategory. For any p and n, $L_n^f \mathcal{T} = L_n^f \mathcal{T}_\varepsilon$. The same holds for L_n since $L_n = L_n L_n^f$.*

Proof. We only have to prove $L_n^f \mathcal{T}_\varepsilon \subseteq L_n^f \mathcal{T}$. Let Σ be an arbitrary finite set of primes, and F be a finite complex such that q-localization of F is not zero and $L_n^f F = 0$.

Then by the definition of ε-enlargement, $\mathcal{T}_{\varepsilon,\Sigma} \subseteq \mathcal{T}_F$. However by definition $L_n^f \mathcal{T}_F = L_n^f \mathcal{T}$ since L_n^f is smashing, hence the result. □

Remark. We can actually strengthen this result by the following two propositions.

Proposition 2.5. *$\mathcal{C} \in \mathrm{Cat}_\infty^{perf}$, $\mathcal{T} \subseteq \mathcal{C}$ be a thick subcategory, then $X \in \mathcal{T}_\varepsilon \iff$ its image in $\mathcal{C}/\mathcal{T}$ is an ε-object.*

Theorem 2.6. *$\mathcal{C} \in \widehat{\mathrm{Cat}_\infty^{perf}}$, $X \in \mathcal{C}$ is an ε-object if and only if the following equivalent conditions holds:*

1. $T(n)_* End_{\mathcal{C}}(X) = 0, \forall n \in [0, \infty), p$.

2. *For any exact functor* $F : \mathcal{C} \to L_{T(n)}Sp$*, the image of* X *is zero.*

The same holds for $K(n)$*.*

In the end of this subsection we remark that ε-enlargement preserves thick $\otimes$-ideal in monoidal presentable stable idempotnet complete ∞-category.

Proposition 2.7. $\mathcal{C} \in \mathrm{CAlg}(\widehat{\mathrm{Cat}_\infty^{perf}})$*, if* $\mathcal{I} \subseteq \mathcal{C}$ *preserves thick* $\otimes$*-ideal* ($X \in \mathcal{I}, Y \in \mathcal{C} \implies X \otimes Y \in \mathcal{I}$)*, then* $\mathcal{I}_\varepsilon$ *is also a thick* $\otimes$*-ideal.*

Proof. It suffices to show that $\forall Y \in \mathcal{C}$, $Y \otimes -$ maps $\mathcal{I}_\varepsilon$ into $\mathcal{I}_\varepsilon$. By theorem 2.3, it suffices to check $Y \otimes -$ maps $\mathcal{I}$ into $\mathcal{I}$ since $Y \otimes -$ is exact. However this is guaranteed by $\mathcal{I}$ is a thick $\otimes$-ideal. □

2.2 ε-nilpotent tower

In this subsection we use the language of ε-tower to build the basic framework in examining the limit diagram computing homotopy fixed point (or more generally descent).

Definition 2.8. Let $\mathcal{C} \in \widehat{\mathrm{Cat}_\infty^{\mathrm{perf}}}$, $\mathrm{Tow}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathcal{C})$. We note that $\mathrm{Tow}(\mathcal{C}) \in \widehat{\mathrm{Cat}_\infty^{\mathrm{perf}}}$ as well.

1. $\mathrm{Tow}^{nil}(\mathcal{C}) \subseteq \mathrm{Tow}(\mathcal{C})$ denote the full subcategory spanned by those towers $\{X_i\}$ such that $\exists N \in \mathbb{Z}_{\geq 0}, \forall i \in \mathbb{Z}_{\geq 0}, X_{i+N} \to X_i$ is null-homotopic. Such tower is called nilpotent.

2. $\mathrm{Tow}^{const}(\mathcal{C})$ denote the thick subcategory generated by $\mathrm{Tow}^{nil}(\mathcal{C})$ and the constant towers. Such tower is called quickly converging.

3. $\mathrm{Tow}^{\varepsilon,nil}(\mathcal{C}) := (\mathrm{Tow}^{nil}(\mathcal{C}))_\varepsilon$.

4. Given an object $X \in \mathcal{C}$ and a tower $\{X_i\} \in \mathrm{Tow}(\mathcal{C})$, we say the map of tower $\{X \to X_i\}$ exhibits X as an ε-nilpotent limit of the tower $\{X_i\}$ if the cofiber tower $\{X_i/X\}$ belongs to $\mathrm{Tow}^{\varepsilon,nil}(\mathcal{C})$.

5. Given an augmented cosimplicial object $X^\bullet \in \mathrm{Fun}(\Delta^+, \mathcal{C})$, we say it is an ε-nilpotent limit diagram if $\{X^{-1} \to \mathrm{Tot}_i(X^\bullet)\}$ exhibits ε-nilpotent limit.

Remark. Directly by theorem 2.3, we have all these properties preserved under exact functors.

The key property of ε-nilpotent diagram is that it becomes actual limit diagram after taking finite localizations.

Proposition 2.9. *Let* $\mathcal{C} \in \mathrm{Cat}_\infty^{perf}$*, suppose* $X^\bullet \in \mathrm{Fun}(\Delta^+, \mathcal{C})$ *is an* ε*-nilpotent limit diagram, then:*

1. $X^{-1} \to \mathrm{Tot}(X^\bullet)$ *is an* ε*-equivalence*

2. $\{L_n^f \operatorname{Tot}_i(X^\bullet)\}$ *is quick converging*

3. *Both maps*

$$L_n^f X^{-1} \to L_n^f \operatorname{Tot}(X^\bullet) \to \operatorname{Tot}(L_n^f X^\bullet)$$

are equivalences.

Proof. We first prove item 1 and 3. Set $\{Y_i\} = \{\operatorname{Tot}_i(X^\bullet)/X^{-1}\}$, it suffices to show that if $\{Y_i\} \in \operatorname{Tow}^{\varepsilon,nil}(\mathcal{C})$, then $\varprojlim Y_i$ is an ε-object, and $L_n^f \varprojlim Y_i \simeq \varprojlim L_n^f Y_i \simeq 0$. (Localization functors are exact).

Now we consider the three functors

$$F_1, F_2, F_3 : \operatorname{Tow}(\mathcal{C}) \to \mathcal{C}$$

$$F_1(\{Y_i\}) = L_n^f \varprojlim Y_i, \quad F_2(\{Y_i\}) = \varprojlim L_n^f Y_i, \quad F_3(\{Y_i\}) = \varprojlim Y_i$$

By theorem 2.3, the image of F_1, F_2, F_3 lies in the corresponding ε-enlargement. For F_1, F_2, since the image are both L_n^f-local, theorem 2.4 furthermore guarantee that these are zero.

For item 2, it suffices to show that if $\{Y_i\} \in \operatorname{Tow}^{\varepsilon,nil}(\mathcal{C})$, then $\{L_n^f Y_i\} \in \operatorname{Tow}^{nil}(\mathcal{C})$(since Y_i is obtained by quotienting a constant tower). Again theorem 2.4 leads to the result. □

Now we discuss about a certain type of tower: cobar complexes.

Definition 2.10. Let $\mathcal{C} \in \operatorname{CAlg}(\widehat{\operatorname{Cat}_\infty^{\mathrm{perf}}})$, define Nil^A the thick $\otimes$-ideal generated by A, we call an object in it a A-nilpotent object.

Define $\operatorname{Nil}^{A,\varepsilon} = (\operatorname{Nil}^A)_\varepsilon$ be its ε-enlargement and call this the subcategory of (A, ε)-nilpotent objects.

The key property of (A, ε)-nilpotence is the following:

Theorem 2.11. *Let* $\mathcal{C} \in \operatorname{CAlg}(\widehat{\operatorname{Cat}_\infty^{perf}})$, $A \in \operatorname{Alg}(\mathcal{C})$. *If* $X \in \mathcal{C}$ *is* A*-nilpotent, then* $CB^\bullet_{aug}(A) \otimes X$ *is a limit diagram, furthermore the associated tower* $\{\operatorname{Tot}_i(CB^\bullet(A) \otimes X)/X\}$ *is nilpotent.*

Proof. Consider the subcategory $\mathcal{I}$ consists of all X such that $\{\operatorname{Tot}_i(CB^\bullet(A) \otimes X)/X\}$ is nilpotent. $\mathcal{I}$ is automatically $\otimes$-ideal since $M \to N$ null-homotopic $\implies$ $M \otimes P \to N \otimes P$ null-homotopic.

We prove that $\mathcal{I}$ is also thick, this is because we only need to check closeness under cofiber, translation and retract. For cofiber this is because

$$\operatorname{Tot}_i(CB^\bullet(A) \otimes (Y/X))/(Y/X) = \frac{\operatorname{Tot}_i(CB^\bullet(A) \otimes Y)/Y}{\operatorname{Tot}_i(CB^\bullet(A) \otimes X)/X}$$

for translation this is obvious. For retract: suppose $Y \to X \to Y$ is id_Y, i.e. Y is retract of X, then the map

$$\operatorname{Tot}_i(CB^\bullet(A) \otimes Y)/Y \to \operatorname{Tot}_i(CB^\bullet(A) \otimes X)/X \to \operatorname{Tot}_i(CB^\bullet(A) \otimes Y)/Y$$

is also an retract. However retract of an zero morphism is zero, since one can check the diagram:

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \big\downarrow{\scriptstyle i} & & {\scriptstyle i}\big\downarrow \\ \bullet & \xrightarrow{0} & \bullet \\ \big\downarrow{\scriptstyle r} & & {\scriptstyle r}\big\downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Now we show that $\mathcal{I}$ is thick $\otimes$-ideal, it suffices to prove $A \in \mathcal{I}$. However this is true since by direct computation, we have:

$$\mathrm{Tot}_n(CB^\bullet(A) \otimes A) = \mathrm{cofib}(\bar{A}^{\otimes(n+1)} \otimes A \to A)$$

where $\bar{A} = \mathrm{fib}(\mathbf{1} \to A)$. (For detailed computation of this totalization, we refer to [MNN17, Section 2.1])

Hence by the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \bar{A}^{\otimes(n+1)} \otimes A & \longrightarrow & \bar{A}^{\otimes(n+1)} \otimes A \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \longrightarrow & A & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \longrightarrow & \mathrm{Tot}_n(CB^\bullet(A) \otimes A) & \longrightarrow & \mathrm{Tot}_n(CB^\bullet(A) \otimes A)/A \end{array}$$

we have

$$\mathrm{Tot}_n(CB^\bullet(A) \otimes A)/A \simeq \Sigma(\bar{A}^{\otimes^{n+1}} \otimes A)$$

However $\bar{A} \otimes A \to A$ is the fiber of $A \to A \otimes A$, while this map has a section, hence the fiber must be null-homotopic, hence we get the result. □

Corollary 2.12. *Let $\mathcal{C}, A$ same as previous theorem, suppose X is (A, ε)-nilotent, then the augmented cosimplicial object $CB^\bullet(A)_{aug} \otimes X$ is an ε-nilpotent limit diagram, here the augmentation of $CB^\bullet(A)$ at point (-1) is given by the tensor unit $\mathbf{1}$.*

Proof. One consider the exact functor $\mathcal{C} \to \mathrm{Tow}(\mathcal{C})$:

$$X \mapsto \{\mathrm{Tot}_i(CB^\bullet(A) \otimes X)/X\}$$

Previous theorem shows that this functor maps Nil^A into Tow^{nil}, hence by theorem 2.3 it maps $\mathrm{Nil}^{A,\varepsilon}$ into $\mathrm{Tow}^{nil,\varepsilon}$. □

3 General results

First we prove the following result to show a object is a (R, ε)-nilpotent, which can be furthermore used to deduce the descent result.

Theorem 3.1. *Let $\mathcal{C} \in \mathrm{CAlg}(\widehat{\mathrm{Cat}_\infty^{perf}})$, $R \in \mathrm{CAlg}(\mathcal{C})$, suppose exists $M \in \mathcal{C}$ and map $M \to R$ in $\mathcal{C}$ such that the image $\pi_0(M) \otimes \mathbb{Q} \to \pi_0(R) \otimes \mathbb{Q}$ contains the unit, then $R \in Nil^{M,\varepsilon}$*

Proof. Let N be a positive integer such that $(\pi_0 M)[N^{-1}] \to (\pi_n R)[N^{-1}]$ with image containing the unit. Let Σ be the set of prime factors of N. We prove that $R \in (\mathrm{Nil}^M)_{\varepsilon,\Sigma}$.

Fix any finite spectra F such that F has non-trivial localization $\forall p \in \Sigma$. It suffices to show R lies in the thick $\otimes$-ideal generated by M and $F \otimes \mathbf{1}$. We denote this ideal by $\mathcal{J}$.

Hence it's equivalent to show $\bar{R} \in \mathcal{C}/\mathcal{J}$ is zero, hence equivalent to show $\mathbb{E}_\infty$-ring spectra $B = \mathrm{Hom}_{\mathcal{C}/J}(\mathbf{1}, \bar{R})$ is zero.

Now that the condition ensures exists $\mathbf{1} \to M$ such that its composition with $M \to R$ is $N^k \in \pi_0(R)$. Passing to $\mathcal{C}/\mathcal{J}$, since M is mapped to zero, this shows that in $\pi_0 B$,$N^k = 0$.

This shows that for $p \nmid N$, $H\mathbb{F}_p \otimes B = 0$ and $H\mathbb{Q} \otimes B = 0$ (since N^k is unit in $H\mathbb{F}_p, H\mathbb{Q}$ homology).

Now if $p|N$,

$$B \otimes F = \mathrm{Hom}_{\mathcal{C}/J}(\mathbf{1}, \bar{R}) \otimes F = \mathrm{Hom}_{\mathcal{C}/\mathcal{J}}(\mathbf{1}, F \otimes \bar{R}) = 0$$

However F has nontrivial $\mathbb{F}_p$ homology, while $H\mathbb{F}_{p_*}(B \otimes F) = 0$, by Kunneth $H\mathbb{F}_p \otimes B = 0$

Previous discussion shows that $H\mathbb{Z} \otimes B = 0$, by nilpotence theorem for $\mathbb{E}_\infty$-ring spectra, this shows $\pi_* B$ is nilpotent. However this implies $1 \in \pi_* B$ is nilpotent hence zero, hence the result. ☐

Corollary 3.2. *Let $\mathcal{C} \in \mathrm{CAlg}(\widehat{\mathrm{Cat}_\infty^{perf}})$, $R \in \mathrm{CAlg}(\mathcal{C})$, $A \in \mathrm{Alg}(\mathcal{C})$. Suppose exists A-module M and a map $M \to R$ in $\mathcal{C}$ such that the image of $\pi_0 M \otimes \mathbb{Q} \to \pi_0 R \otimes Q$ contains the unit, then $R \in Nil^{A,\varepsilon}$, hence $CB^\bullet_{aug}(A) \otimes R$ is an ε-nilpotent limit diagram.*

To put the previous result in practice, since we're dealing with K-theory, we actually don't need to remember the entire information of category of perfect modules, but only its additive invariant. The framework is built in [BGT13] and its generalization on $\mathcal{R}$-linear case.

Definition 3.3 (Additive invariant)**.** Given $\mathcal{R} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$, we say a functor is $\mathcal{R}$-linear additive invariant if it sends 0 to 0, split exact sequence to cofiber sequence.

Definition-Theorem 3.4. *There exists a presentable symmetric monoidal stable ∞-category $\mathcal{M}_\mathcal{R}^{add}$ satisfying the following property:*

There exists an additive invariant $\mathcal{U}_{add} : \mathrm{Mod}_\mathcal{R}(\mathrm{Cat}_\infty^{perf}) \to \mathcal{M}_\mathcal{R}^{add}$ and an equivalence

$$\mathrm{Fun}^L(\mathcal{M}_\mathcal{R}^{add}, \mathcal{D}) \to \mathrm{Fun}^{add}(\mathrm{Mod}_\mathcal{R}(\mathrm{Cat}_\infty^{perf}), \mathcal{D})$$

for any presentable stable $\mathcal{D}$, where the equivalence is induced by $\mathcal{U}_{add}$. Fun^L denotes cocontinuous functors and Fun^{add} denotes additive invariants.

Remark. For set-theoretic issue, we always assume that when saying the module category $\mathrm{Mod}_\mathcal{R}(\mathrm{Cat}_\infty^{\mathrm{perf}})$, we are actually talking about the κ-compact object of the module category, where κ is a regular cardinal.(Hence the κ-compact object is closed under pushouts, retracts, and $\mathcal{R}$-linear tensor) In practice, we always assume that κ is large enough.

Definition-Theorem 3.5. *For $\mathcal{C} \in \mathrm{Mod}_{\mathcal{R}}(\mathrm{Cat}_\infty^{perf})$ we define the $\mathcal{R}$-linear (connective) K-theory*

$$K^{\mathcal{R}}(\mathcal{C}) = Map_{\mathcal{M}_{\mathcal{R}}^{add}}(\mathcal{U}^{add}(\mathcal{R}), \mathcal{U}^{add}(\mathcal{C})) \in \mathrm{Sp}$$

Moreover when $\mathcal{R} = \mathrm{Sp}^\omega$, $K^{\mathcal{R}}$ reduced to usual K-theory.

Now we apply these results on $\mathcal{M}_{\mathcal{R}}^{add}$.

Theorem 3.6. *Let $\mathcal{R} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{perf})$, $\mathcal{A} \in \mathrm{Alg}(\mathrm{Mod}_{\mathcal{R}}(\mathrm{Cat}_\infty^{perf}))$. Suppose there exists an $\mathcal{R}$-linear functor $\mathcal{A} \to \mathcal{R}$ whose image on $K_0(-) \otimes \mathbb{Q}$ contains a unit, then the augmented cosimplicial object in $\mathcal{M}_{\mathcal{R}}^{add}$*

$$\mathcal{U}_{add}(\mathcal{R}) \to (\mathcal{U}_{add}(\mathcal{A}) \rightrightarrows \mathcal{U}_{add}(\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}) \cdots)$$

is an ε-nilpotent diagram.

Proof. Apply theorem 3.2 with $\mathcal{C} = \mathcal{M}_{\mathcal{R}}^{add}$, $R = \mathcal{U}_{add}(\mathcal{R})$ and $M = A = \mathcal{U}_{add}(\mathcal{A})$. We only have to verify the map $(\pi_0 M \to \pi_0 R)$ is rationally epi.

However this is exactly saying $K_0^{\mathcal{R}}(\mathcal{A}) \to K_0^{\mathcal{R}}(\mathcal{R})$ is rationally epi. By a comparison theorem we do have $K_0^{\mathcal{R}} \cong K_0$, hence the result.

□

4 Examples and applications

First we state the most basic example.

Theorem 4.1. *Let A an $\mathbb{E}_\infty$-ring, B an $\mathbb{E}_2$-algebra in the ∞-category of A-modules. Suppose B is a perfect A module and the map $K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ has image containing the unit.*

Suppose F is an additive invariant of $\mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}_\infty^{perf})$, then the augmented cosimplicial diagram

$$F(\mathrm{Perf}(A)) \to (F(\mathrm{Perf}(B)) \rightrightarrows F(\mathrm{Perf}(B \otimes_A B)) \cdots)$$

is an ε-nilpotent limit diagram. As a consequence the associated Tot tower is quickly convergent after L_n^f.

Proof. Apply theorem 3.6 with $\mathcal{R} = \mathrm{Perf}(A), \mathcal{A} = \mathrm{Perf}(B)$. ($B$ is $\mathbb{E}_2$ ensures the perfect module category is monoidal). The $\mathcal{R}$-linear functor $\mathcal{A} \to \mathcal{R}$ is given by forgetful functor.

[HA, Section 4.8.5] ensures that the functor from A-algebra to $\mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}_\infty^{\mathrm{perf}})$ given by $B' \mapsto \mathrm{Perf}(B')$ is monoidal, hence we have $\mathrm{Perf}(B \otimes_A \otimes B) \simeq \mathrm{Perf}(B) \otimes_{\mathrm{Perf}(A)} \otimes \mathrm{Perf}(B)$. Hence the results naturally follows since F factor through an exact functor (since $\mathcal{D}$ is stable), and exact functor preserve ε-nilpotent limit diagram by section 2.2thesubsection. □

Now we examine the descent result in some concrete settings.

4.1 Fppf descent for discrete ring

Proposition 4.2. *Let $A \to B$ be a morphism of $\mathbb{E}_\infty$-ring spectra, suppose:*

1. *$\pi_0(B)$ is a faithfully flat, finite and projective $\pi_0(A)$-module*

2. *The canonical map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism*

Then the hypotheses of theorem 4.1 is satisfied.

Proof. B is a finitely generated projective A-module (hence perfect) comes from the following technical lemma:

Lemma 4.3. *Let A be an $\mathbb{E}_1$-ring, and M an A-module such that $\pi_0(M)$ is projective over A; $\pi_*(M) \cong \pi_*(A) \otimes_{\pi_0(A)} \pi_0(M)$. Then M is projective over A. Moreover if $\pi_0 M$ is finitely generated, then M is finitely generated projective A module.*

To show that the hypotheses of theorem 4.1 is satisfied, we note that if $X \to Y$ satisfies the condition, then so is $X' \otimes Y \otimes_X X'$. Hence it suffices to prove $\tau_{\geq 0} A \to \tau_{\geq 0} B$ satisfies the condition since $B \simeq A \otimes_{\tau_{\geq 0} A} \tau_{\geq 0} B$.

Hence we assume A, B are both connective itself. In this case, $K_0(A) = K_0(\pi_0 A)$ by [Lura, Lecture 20, Corollary 3]. Hence the problem is now reduced to a pure classical algebraic problem, i.e. given two (ordinary) ring $R \to S$ such that S is faithfully flat, finite and projective R-module, we have to check $[S] \in K_0(R) \otimes \mathbb{Q}$ is a unit.

Since Q is faithfully flat finite projective module, $\dim_{\kappa(\mathfrak{p})} S \otimes_R \kappa(\mathfrak{p})$ is a constant, say n. Moreover one can find a finite open cover $X = \cup_{i=1}^r U_i$, such that $x := [P] - n$ vanishes in $K_0(U_i)$. ([Stack, Lemma 10.78.2])

Hence by Zariski descent for algebraic K-theory, consider the sequence:

$$K_1(U_1 \cap U_2) \to K_0(R) \to K_0(U_1) \oplus K_0(U_2)$$

Since $x, x^2 \in K_0(R)$ has zero image in $K_0(U_1) \oplus K_0(U_2)$, both of the elements has a preimage in $K_1(U_1 \cap U_2)$, say $a, x \cdot a$. However the action of $K_0(R)$ on $K_1(U_1 \cap U_2)$ comes from its restriction to $K_0(U_1 \cap U_2)$, hence $x \cdot a = 0 \in K_1(U_1 \cap U_2)$. As a result $x^2 = 0 \in K_0(R)$.

For general open covering given by r open sets, we can deduce by induction that $x^r = 0 \in K_0(R)$.

Hence $[P] = n(1 + \dfrac{[P] - n}{n})$ is a rational unit, since $1 + \dfrac{[P] - n}{n}$ is the inverse of $1 + \sum_r \dfrac{(n - [P])^r}{n^r}$. □

Corollary 4.4. *When restricted to discrete rings, localized algebraic K-theory satisfies fppf descent.*

Proof. By [Stack, Section 37.48], one suffices to check descent property with respect to Zariski cover and surjective finite locally free morphisms, which are guarenteed by the previous proposition. □

4.2 Galois descent for $\mathbb{E}_\infty$-ring

In this subsection, we restrict our sight only to Galois extension.

Theorem 4.5. *Let $A \to B$ a G-Galois extension of $\mathbb{E}_\infty$-ring spectra (in the sense of [Rog08]) where G is finite. Suppose the image of the map $K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ contains the unit, then*

1. *$K(A) \to K(B)^{hG}$ is an ε-equivalence*
2. *The tot-tower computing $L_n^f K(B)^{hG}$ is quickly converging*
3. *Both maps*
$$L_n^f K(A) \to L_n^f(K(B)^{hG}) \to (L_n^f K(B))^{hG}$$
are equivalences

Proof. B is dualizable by [Rog08, Proposition 6.2.1], hence perfect. By theorem 4.1, the cobar construction $B \otimes_A B \otimes \cdots$ turns into resolution computing homotopy fixed point, since $B \otimes_A B \simeq \prod_{g \in G} B$.

On the other hand $B \mapsto \mathrm{Perf}(B) \mapsto \mathcal{U}_{add}(\mathrm{Perf}(B)) \mapsto K(\mathrm{Perf}(B))$ preserves finite products, hence the result. □

Corollary 4.6. *The descent spectral sequence computing $L_n^f K(A)$ has a horizontal vanishing line.*

Proof. This is auotmatic by the fact that tot-tower is quickly converging, hence the filtration index is bounded. □

We now take a closer look at the technical condition ensuring Galois descent:

Lemma 4.7. *Let $A \to B$ a G-Galois extension of $\mathbb{E}_\infty$-ring spectra where G is finite, then the following are equivalent:*

1. *$K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ contains a unit*
2. *$[B] \in K_0(A) \otimes \mathbb{Q}$ is a unit*
3. *$[B] = |G| \in K_0(A) \otimes \mathbb{Q}$*

Proof. $3 \implies 2 \implies 1$ is trivial.

$1 \implies 3$. Consider $i^* : K_0(A) \otimes \mathbb{Q} \to K_0(B) \otimes \mathbb{Q}$ and $i_* : K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$. Say $x \in K_0(B) \otimes \mathbb{Q}$ s.t. $i_*(x) = 1$. However we have:

$$i^* \circ i_*([M]) = [M \otimes_A B] = [M \otimes_B B \otimes_A B] = [M \otimes_B \prod_G B] = |G| \cdot [M]$$

Hence in $K_0(B)$, $|G| \cdot x = i^* \circ i_*(x) = i^*(1) = [B]$. Hence in $K_0(A)$, $[B] = |G|$. □

We say a G-Galois extension satisfies **condition A** if it satisfies the previous equivalent statements.

Proposition 4.8. *Consider the C_2-Galois extension $KO \to KU$. It satisfies condition A and $K(KO) \to K(KU)^{hC_2}$ is a ε-equivalence.*

Proof. Consider $[KU] \in K_0(KO)$, by wood's theorem $KO \otimes \Sigma^{-2}\mathbb{C}P^2 \simeq KU$. Since $[\Sigma^{-2}\mathbb{C}P^2] = [\mathbb{S}^0 \cup \mathbb{S}^2] = 2 \in K_0(\mathbb{S})$, $[KU] = 2 \in K_0(KO)$, hence the condition A is satisfied, the rest naturally follows. □

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